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EQUIVELAR POLYHEDRAL MANIFOLDS IN E^3

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ABSTRACT

An equivelar polyhedral 2-manifold in the class $\mathcal{M}_{p,q}$ is one embedded in E^3 in which every face is a convex p-gon and every vertex is q-valent. Various constructions for equivelar manifolds are described, and it is shown that, in certain classes $\mathcal{M}_{p,q}$, there is a manifold of given genus $g \geqq 2$ for all but finitely many g.

§1. Introduction

A *polyhedral manifold*, or, more briefly (and only for the purposes of this paper), a *polyhedron*, is a closed topological manifold in some euclidean space, which is the underlying point-set of a geometric 2-complex (in the sense of [13], §3.2), whose faces (that is, 2-cells) are convex polygons. (We similarly refer to the 0- and 1-cells as vertices and edges.) Thus, roughly speaking, a polyhedron is a geometric model of an abstract 2-manifold, made up of convex polygons.

In this paper, we shall be concerned primarily with polyhedra in ordinary space E^3; such polyhedra are necessarily orientable. In accordance with our intuitive feelings, we shall further require that adjacent faces (that is, faces which share a common edge) not be coplanar; from a theoretical viewpoint, this requirement may strictly be unnecessary.

We say a polyhedron M is *equivelar* if each of its faces has the same number p of edges, and if each of its vertices belongs to the same number q of edges. The family of combinatorial isomorphism classes of such polyhedra we denote by $\mathcal{M}_{p,q}$, and a particular such manifold we write as $\{p, q\}$, or as $\{p, q; g\}$ if we wish also to note its corresponding genus g.

The introduction of the new term equivelar needs some justification. As the definition implies, an equivelar polyhedron M has local regularity properties. In case M is a (topological) sphere, it is easy to see that M must be combinatorially regular, in that the combinatorial automorphism group of M is transitive on the

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flags of M, which are the triples $\{F^0, F^1, F^2\}$, where F^i is an i-cell of M, with $F^0 \subset F^1 \subset F^2$. (Compare [14].) But for higher genus this is, in general, not true — equivelarity is a weaker condition than (combinatorial) regularity. Certain of the polyhedra we discuss below will, in fact, be combinatorially regular, but we shall place no particular emphasis on this. Our notation $\{p, q; g\}$ is, of course, a simple modification of the Schläfli–Coxeter symbol for regular polyhedra (see [7]).

Before we come to our results, we mention that there is a considerable literature on geometric realizations of 2-manifolds in E^3 or E^4; we single out [1; 2; 3; 4; 12; 19] as being more particularly relevant to the present paper. There is also an extensive literature on regular polytopes and related topics; see, for example, [7; 8; 10; 17].

We also note the various infinite analogues of our equivelar polyhedra, which we might call equivelar apeirohedra; in an obvious extension of our notation, we can denote these by $\{p, q; \infty\}$. There are the regular Petrie–Coxeter skew-polyhedra $\{4, 6; \infty\}$, $\{6, 4; \infty\}$ and $\{6, 6; \infty\}$ ([6]), and other examples with regular faces due to Gott [11], namely $\{3; 8; \infty\}$, $\{3, 10; \infty\}$, $\{4, 5; \infty\}$ and $\{5, 5; \infty\}$ (some of these actually violate our condition that adjacent faces not be coplanar); compare also [19]. $\{5, 5; \infty\}$ and $\{6, 6; \infty\}$ are noteworthy, in that they are the only equivelar manifolds $\{p, q\}$ (finite or infinite) of which we are aware, with both $p > 4$ and $q > 4$. We have also found (by methods analogous to those described below) many other examples, but whether Theorem 1 or 2 generalizes to the case $g = \infty$ we do not know. However, although the local appearance of these apeirohedra is like that of our (compact) equivelar polyhedra, their global behaviour is quite different; for example, they are usually invariant under a certain group of lattice translations.

§2. Statement of results

For equivelar polyhedra, very simple combinatorial conditions hold. Let $M \in \mathcal{M}_{p,q}$ be a polyhedron of genus g, and denote by $f_i = f_i(M)$ the number of its i-cells ($i = 0, 1, 2$). Then we have

$$qf_0 = 2f_1 = pf_2.$$

From this and Euler's equation

$$f_0 - f_1 + f_2 = 2 - 2g$$

follows

$$\frac{1}{2} - \frac{1}{p} - \frac{1}{q} = \frac{2}{q} \cdot \frac{g-1}{f_0} = \frac{g-1}{f_1} = \frac{2}{p} \cdot \frac{g-1}{f_2}. \tag{1}$$

Relation (1) leads to an easy classification.

(a) If $1/p + 1/q > \frac{1}{2}$, then $g = 0$ and $\mathcal{M}_{p,q}$ consists of (the combinatorial isomorphism class of) the single regular convex polyhedron $\{p, q\}$; that is, the Platonic solids $\{3,3\}$, $\{3,4\}$, $\{3,5\}$, $\{4,3\}$ and $\{5,3\}$.

(b) If $1/p + 1/q = \frac{1}{2}$, then $g = 1$ and $\mathcal{M}_{p,q}$ can consist of tori alone. The classes $\mathcal{M}_{3,6}$ and $\mathcal{M}_{4,4}$ contain infinitely many such tori, but, of course, $\mathcal{M}_{6,3}$ is empty, because a polyhedron of genus $g \geqq 1$ must have some non-convex vertices, which is impossible if all vertices are 3-valent ([13], exercise 11.1.7).

(c) If $1/p + 1/q < \frac{1}{2}$, then $g \geqq 2$, and g increases with f_0, f_1 and f_2. Again, the classes $\mathcal{M}_{p,3}$ ($p \geqq 7$) must be empty.

From (1) it follows that only certain genera g can occur in a given class $\mathcal{M}_{p,q}$. In 19 classes, all but finitely many values of g are allowed by (1). For this to be possible, $(2/p)(\frac{1}{2} - 1/p - 1/q)^{-1}$ and $(2/q)(\frac{1}{2} - 1/p - 1/q)^{-1}$ must both be integers. We list in Table 1 these classes, with the smallest possible value of g; however, the easy computations which lead to this value (they involve counting edges and diagonals of faces) are omitted.

Table 1

$p \backslash q$	4	5	6	7	8	9	10	12	18
3				2	3	4	5	8	20
4		3	5		10			26	
5	3	5					32		
6	5		14						
8	10				50				
10		32							
12	26								

In fact, Betke and Gritzmann [5] have shown that, if $q \geqq 5$ is odd, then $\mathcal{M}_{p,q}$ is empty unless $p < 2q$; hence the entry $\{10, 5\}$ in Table 1 should be omitted.

If $g \geqq 2$, equivelar manifolds have the nice property that they have the minimal number of vertices, edges and faces among all polyhedra of genus g with at most p-gonal faces and at most q-valent vertices. This follows directly from (1) (using $qf_0 \geqq 2f_1$, $pf_2 \geqq 2f_1$). This property was a starting point for our research, and we began with the investigation of some of the 19 classes mentioned above (especially those with small p and q). However, the relative difficulty of some of the constructions needed soon made clear that it was an interesting question as to which of the classes $\mathcal{M}_{p,q}$ were infinite, or even non-empty.

Further developments of the constructions we employed (such as inscribing manifolds and intersecting octahedra, which we describe below) led to infinite numbers of infinite classes $\mathcal{M}_{p,q}$. These wider constructions are discussed in another paper [15], one of whose results is:

THEOREM 1. *Each of the classes* $\mathcal{M}_{3,q}$ $(q \geqq 7)$, $\mathcal{M}_{4,q}$ $(q \geqq 5)$ *and* $\mathcal{M}_{p,4}$ $(p \geqq 5)$ *is infinite.*

In this paper, we shall largely concentrate on the problem of realizing all possible genera within given classes $\mathcal{M}_{p,q}$. Our main results are:

THEOREM 2. *There exist the following equivelar polyhedra*:
(a) $\{3,7;g\}$ *for* $g \geqq 2$;
(b) $\{3,8;g\}$ *for* $g \geqq 4$;
(c) $\{4,5;g\}$ *and* $\{5,4;g\}$ *for* $g = 5,7$ *and* $g \geqq 9$;
(d) $\{3,9;g\}$, $\{4,6;g\}$ *and* $\{6,4;g\}$ *for* $g = 6,9,10$ *and* $g \geqq 12$.

In Theorem 2, we have listed all the classes $\mathcal{M}_{p,q}$ in which we have been able to find polyhedra $\{p,q;g\}$ with all except possibly finitely many of the genera g which can occur. So, we have only mentioned 7 of the 19 classes of Table 1; in $\mathcal{M}_{3,7}$ we have all possible manifolds, in $\mathcal{M}_{3,8}$, one is missing, in $\mathcal{M}_{3,9}$, 5 are missing, and in the remaining four classes, 4 are missing. It is probable, though, that for geometric (rather than combinatorial) reasons, some of these missing manifolds do not, in fact, exist.

The constructions used to prove Theorem 1 give a fairly sparse set of genera. However, those used to prove Theorem 2 (see Lemmas 5, 8a, 9a, 14, 15, 16 and 17) can be applied in other circumstances, and yield the following additional polyhedra (Lemmas 7, 8b and 9b):

(e) $\{3,12;g\}$, $\{4,8;g\}$ *and* $\{9,4;g\}$ *for* $g = 73 + 66n$, $n \geqq 0$.

All these results are illustrated in Fig. 1. In this figure, the Platonic solids and plane tessalations (and related tori) are indicated by P resp. T, Coxeter's $\{6,6;\infty\}$ and Gott's $\{5,5;\infty\}$ by ∞, the classes of Theorem 2 by solid discs, and the remaining classes of Theorem 1 by open circles.

§3. Four-dimensional constructions

If P is any 4-polytope, then a Schlegel diagram of P is obtained by projecting the faces of P radially from a point beyond one facet of P but beneath all the others into that facet. In particular, each 2-face of P is projected into a convex polygon, and incidences between 2-faces are preserved, so that in the Schlegel

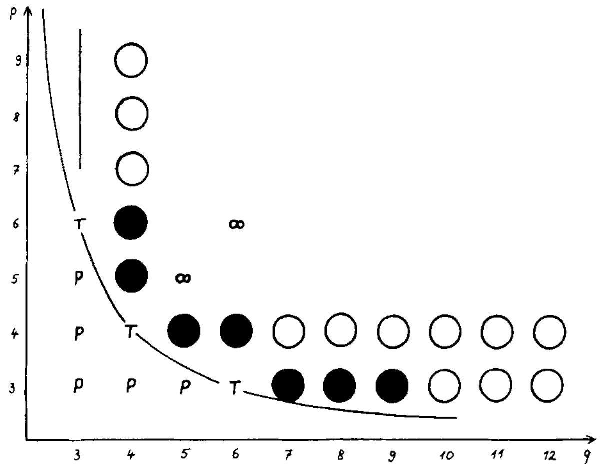

Fig. 1.

diagram is an isomorphic copy of the 2-skeleton of P (see [13] for more details). In other words:

LEMMA 1. *If a 2-manifold M is geometrically realizable in the 2-skeleton of some 4-polytope, then M is geometrically realizable in E^3.*

In what follows, we shall use this lemma without further comment.

Our construction of suitable manifolds will depend upon certain modifications of 4-polytopes, in particular classes. But we begin by describing the modifications in general terms. Here and below, we start with a 4-polytope P, whose facets are labelled F_i, and whose 2-faces are labelled $F_i \cap F_j = F_{ij} = F_{ji}$ (we only use this notation when F_i and F_j are adjacent facets).

The first modification, which we call *method A*, is the following. Within each facet F_i of P is inscribed a 3-polytope G_i, such that

(a) for each 2-face F_{ij} of F_i, $G_{ij} = G_i \cap F_{ij}$ is a 2-face of G_i;

(b) $G_{ij} = G_{ji}$;

(c) $G_{ij} \subsetneq \operatorname{relint} F_{ij}$.

We now let $Q = \operatorname{conv}(\bigcup G_i)$, so that Q is a 4-polytope, and each G_i is a facet of Q. (Of course, Q will have other facets, but these do not concern us.)

LEMMA 2. *Let Q be as above, and let M consist of those 2-faces of Q which are contained in exactly one facet G_i. Then M is an orientable 2-manifold.*

If we observe that two 2-faces of M belonging to different G_i's can only meet in the boundary of some 2-face G_{ij}, then the lemma is obvious.

Before we describe particular applications, let us calculate the genus of M. For each i, $G_i \cap M$ consists of a sphere from which $f_2(F_i)$ polygons have been removed. Since single circuits can be omitted or not in calculating the Euler characteristic, we obtain

$$\chi(M) = \sum \chi(G_i \cap M)$$

$$= \sum (2 - f_2(F_i))$$

$$= 2f_3(P) - 2f_2(P),$$

the latter term arising since each 2-face of P belongs to two facets. In other words:

LEMMA 3. *Let M be obtained from the 4-polytope P by method* A. *Then*

$$g(M) = 1 + f_2(P) - f_3(P).$$

In particular, let P have facets all isomorphic to the regular 3-polytope $\{r, s\}$ $(=\{r, s; 0\})$. In this case, $f_2(F_i)$ is the constant

$$f_2(\{r, s\}) = \frac{4s}{4-(r-2)(s-2)}$$

(see [7]). We then conclude (using the second expression for $\chi(M)$ above, rather than the last):

LEMMA 4. *Let P be a 4-polytope, all of whose facets are isomorphic to the regular 3-polytope $\{r, s\}$, and let M be obtained from P by method* A. *Then*

$$g(M) = 1 + d_{rs} f_3(P),$$

where

$$d_{rs} = \frac{r(s-2)}{4-(r-2)(s-2)}.$$

In Table 2, we list the values of d_{rs}.

Table 2

r \ s	3	4	5
3	1	3	9
4	2		
5	5		

After this general discussion, we now come to specific examples.

Firstly, let P be a simplicial 4-polytope; that is, every facet of P is a tetrahedron, isomorphic to $\{3,3\}$. In fact, of course, every facet of P is affinely equivalent to $\{3,3\}$. So, if we inscribe in our fixed regular tetrahedron $S = \{3,3\}$ a polyhedron T, such that

(i) every symmetry of S is a symmetry of T;

(ii) for each 2-face H of $S, H \cap T$ is a 2-face of T; and

(iii) $H \cap T \subseteq \operatorname{relint} H$;

and in each facet of P take the appropriate affine image of T (under the affinity carrying S into that facet), we see that we shall have satisfied the conditions (a)–(c) of method A.

For our first example, we take T to be the archimedean truncated tetrahedron with its triangular faces lying in the faces of $\{3,3\}$. Since the vertices of the resulting manifold M are all alike, and belong to four hexagons each, in view of Lemmas 2 and 4, we see that we have

LEMMA 5. *There is a* $\{6,4;g\}$ *for each* $g = 6, 9, 10$ *and* $g \geqq 12$.

For the assertion about the genus, we refer to [13], figure 10.4.1, where it is shown that there are simplicial 4-polytopes with f_3 facets, for $f_3 = 5, 8, 9$ and $f_3 \geqq 11$.

We now apply method A to the 4-polytopes whose facets are all isomorphic to the octahedron $\{3,4\}$. In [16] is described the only construction of which we are aware that yields an infinite class of such 4-polytopes. Since we need a certain feature of this construction, we sketch a description of it here. Our basic building block is the regular 24-cell $B = \{3,4,3\}$; as its name suggests, B has 24 facets, which are all regular octahedra.

Suppose that P_1 is a 4-polytope, all of whose facets are projectively equivalent to regular octahedra. Let F be any such facet, and p a point beyond F, but beneath every other facet of P_1. We can perform a suitable projective mapping on B to obtain B_1, with the properties that $B_1 \subseteq \operatorname{conv}(F \cup \{p\})$, and one facet of B_1 coincides with F. Then $P_2 = P_1 \cup B_1$ is another 4-polytope of the type required. Initially, of course, we have $P_1 = B$ itself.

We observe two things about this construction. Firstly, since $f_3(P_2) = f_3(P_1) + 22$, the only numbers of facets obtainable in this way are $24 + 22n$, $n \geqq 0$. Secondly, not only are all the facets of such polytopes projectively equivalent (to a regular octahedron), but the projective equivalences are consistent on common 2-faces to adjacent octahedra. What we mean by this can be explained as follows. Let two adjacent such octahedra have common vertices a, b, c and remaining vertices a', b', c' and a'', b'', c'', with a', a'' opposite a in their respective octahedra, and so on. Then there is a projectivity of E^4, actually unique, which interchanges the octahedra, leaving a, b, c fixed, and so interchanging a' and a'', and so on; this projectivity leaves every point of the triangle conv$\{a, b, c\}$ fixed.

For our application of method A, we inscribe in a fixed regular octahedron an archimedean truncated cube, whose triangular faces lie in the triangles of the octahedron. The truncated cube then has the same symmetries as the octahedron. Under the projective mappings taking this fixed octahedron into the octahedral facets of our 4-polytope P, these triangular faces of the truncated cubes then coincide in pairs, thus fulfilling the conditions (a)–(c) of method A. So, since four octagons meet at each vertex of the manifold M, we have:

LEMMA 6. *There is a* $\{8,4;g\}$ *for each* $g = 73 + 66n$, $n \geqq 0$.

The genus $73 + 66n$ arises from Lemma 4 and the remark above, since we have $d_{34} = 3$ (Table 2).

We could also apply method A to the two classes of 4-polytopes whose facets are all (projectively regular) cubes or dodecahedra; however, we obtain no new classes of manifolds, and gaps in the numbers of possible genera.

We now move on to *method B*. Again, we first describe the construction in general terms. Let P be a 4-polytope, with facets F_i, as before. Within each facet F_i is inscribed a 3-polytope G_i, such that, for each 2-face F_{ij} of F_i, $G_{ij} = G_i \cap F_{ij}$ is a 2-face of G_i. (We impose no other conditions; notice that $G_i = F_i$ is allowed.) For each i, let $p_i \in \operatorname{relint} G_i$, and let $0 < \lambda < 1$. Let $H_i = (1-\lambda)p_i + \lambda G_i$, $H_{ij} = (1-\lambda)p_i + \lambda G_{ij}$, and let $Q = \operatorname{conv}(\bigcup H_i)$. Among the facets of Q are the H_i, and facets of the form K_{ij} $(= K_{ji}) = \operatorname{conv}(H_{ij} \cup H_{ji})$; for H_{ij} and H_{ji} lie in parallel 2-flats, and so K_{ij} lies in a hyperplane, which clearly supports Q. Finally, our manifold M is formed of those 2-faces of Q which belong to exactly one facet H_i or K_{ij}.

We first observe, as far as the genus of M is concerned:

LEMMA 7. *Lemmas* 3 *and* 4 *hold, with method* B *instead of method* A.

We first apply method B in the cases where the facets of P are all isomorphic to the regular 3-polytope $\{r, s\}$, and the G_i are just the facets F_i. The only 2-faces of M are then the quadrilateral faces of the prisms K_{ij}, and these meet $2s$ at a vertex. In particular, with $\{r, s\} = \{3,3\}$ and $\{3,4\}$, we obtain:

LEMMA 8. *There are*

(a) $\{4,6;g\}$ *for* $g = 6, 9, 10$ *and* $g \geqq 12$;

(b) $\{4,8;g\}$ *for* $g = 73 + 66n$, $n \geqq 0$.

Finally, we modify further these manifolds. Each quadrilateral face of the prisms K_{ij} can be split into two triangles, so that three of these triangles meet at each vertex of K_{ij} (see Fig. 2).

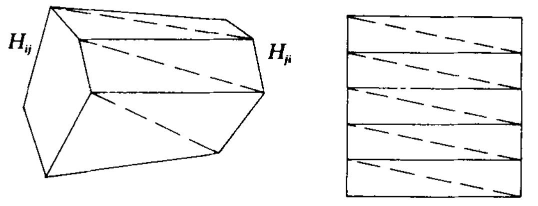

Fig. 2.

This face splitting turns each prism K_{ij} into an antiprism on the same base; furthermore, the face splitting can be achieved geometrically by moving the vertices of M into general position. Thus we have:

LEMMA 9. *There are*
(a) $\{3,9;g\}$ *for* $g = 6,9,10$ *and* $g \geqq 12$;
(b) $\{3,12;g\}$ *for* $g = 73+66n$, $n \geqq 0$.

Let us conclude this section with some remarks. We could also (as with method A) apply method B to those 4-polytopes with cubical or dodecahedral facets, and also to cases with G_i's inscribed in the F_i's, with $G_i \neq F_i$. Further, we could modify existing manifolds by face splittings. However, none of these applications has led us to new classes of manifolds. Indeed, the only ones of any interest are the following.

Firstly, inside each tetrahedron of a simplicial 4-polytope inscribe an icosahedron, and apply method B. The K_{ij} are then triangular antiprisms (octahedra), and the resulting manifold is a $\{3,7;g\}$, with $g = 6,9,10$ and $g \geqq 12$. However, the construction described in Lemma 10 achieves every possible genus. Secondly, we can split each hexagonal face of $\{6,4;g\}$ into four triangles, as in Fig. 3a, so as to turn each truncated tetrahedron used in the construction into an icosahedron (Fig. 3b). There results a $\{3,8;g\}$, with the same range of g just mentioned.

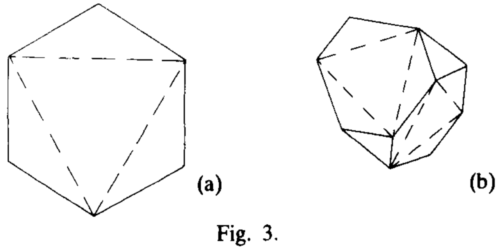

Fig. 3.

We end this section with some observations. If we apply method B with each $G_i = F_i$ to the dual P^* of a given 4-polytope P, we actually obtain an isomorphic polyhedron M. In case P (and P^*) is regular, the process of obtaining the new 4-polytope from P was called by Mrs. Stott *expansion* (see [6]).

It is easily verified that the polyhedra $\{4,6;g\}$ and $\{6,4;g\}$ obtained by methods A and B are actually (combinatorial) dual, as are the pairs $\{4,8;g\}$ and $\{8,4;g\}$.

It is clear from what we have done here that, if we could find 4-polytopes whose facets were all isomorphic to the icosahedron $\{3,5\}$, then we could construct polyhedra $\{10,4;g\}$ (at least if the facets were projectively equivalent to the regular icosahedron), $\{4,10;g\}$ and $\{3,15;g\}$, independently of Theorem 1. However, there are good reasons for supposing that such 4-polytopes do not exist; in any event, it is known that the facets of such a polytope could not all be projectively regular (see [16]).

§4. The method of inscribed manifolds

Our next results are derived from the following constructions, which we first describe combinatorially. Let N be a polyhedral 2-manifold, and $F_1,\cdots,F_k$ disjoint faces of N ($F_i \cap F_j = \emptyset$ if $i \neq j$). Let N' be isomorphic to N, with corresponding faces $F'_1,\cdots,F'_k$. For *method C* (as we shall call it) we delete the pairs of faces F_j and F'_j, and join their corresponding pairs of edges by quadrangles. (So, we have tubular elements, much as in method B.) Thus, at each vertex of a face F_j, F_j is replaced by two quadrangles, and similarly for F'_j. For *method D*, we divide each of these quadrangles into two triangles, keeping the same orientation, as in Fig. 2 above. Thus, at each vertex of F_j, F_j is replaced by three triangles.

LEMMA 10. *If methods* C *or* D *are applied to the faces* $F_1,\cdots,F_k$ *of* N, *then the resulting manifold* M *has*

$$f_0(M) = 2f_0(N), \qquad g(M) = 2g(N) + k - 1.$$

In our applications, the faces $F_1,\cdots,F_k$ will also cover the vertices of N. Two particular cases are of interest here. (In stating the lemma, for simplicity we assume geometric realizability.)

LEMMA 11. *Let the faces* $F_1,\cdots,F_k$ *cover the vertices of* N, *and suppose all the vertices of* N *to be* q*-valent.*

(a) *If all faces of* N, *except possibly* $F_1,\cdots,F_k$, *are quadrangles, and we apply method* C *to obtain* M, *then* $M \in \mathcal{M}_{4,q+1}$.

(b) *If all faces of* N, *except possibly* $F_1, \cdots, F_k$, *are triangles, and we apply method* D *to obtain* M, *then* $M \in \mathcal{M}_{3,q+2}$.

If N is the boundary (complex) of a convex 3-polytope P, the geometric realization of M is easy. For, let the origin 0 of coordinates be an interior point of P, and let $N' = \lambda N$, where $0 < \lambda < 1$. Method C now works geometrically. (Alternatively, observe that M is a subcomplex of the 2-skeleton of the prism over P, and apply Lemma 1. Compare here [18].) For method D, in case all the resulting faces are triangles, we just move the vertices of M into general position, so that the quadrangles split into triangles geometrically.

In general, however, if N is an arbitrary polyhedron in E^3, we must take our isomorphic copy N' in one of the two (open) components of $E^3 \setminus N$. That N' can be suitably so inscribed must then be verified directly.

Before coming to our specific applications of these methods, it is helpful to describe certain modifications of (convex) 3-polytopes which lead to new 3-polytopes.

Let P be a 3-polytope. What we shall call here the *edge polytope* $E(P)$ of P is the 3-polytope with the following facial structure. The vertices of $E(P)$, which are all 4-valent, correspond to the edges of P, and two vertices of $E(P)$ are joined by an edge if the corresponding edges of P meet in a common vertex, and belong to a common face. The faces of $E(P)$ are of two kinds: to an r_i-gonal face of P corresponds an r_i-gonal face of $E(P)$, and to an s_j-valent vertex of P corresponds an s_j-gonal face of $E(P)$. The two kinds of face of $E(P)$ alternate around each vertex. The existence of $E(P)$ as a 3-polytope is guaranteed by Steinitz's theorem ([13], 13.1.1).

Finally, we note that, if P^* is a polytope dual to P, then $E(P^*)$ is isomorphic to $E(P)$, with the rôles of the two kinds of face interchanged. We have $f_0(E(P)) = f_1(P)$, $f_1(E(P)) = 2f_1(P)$ and $f_2(E(P)) = f_0(P) + f_2(P)$.

We shall write $E^2(P) = E(E(P))$, which we call the *second edge polytope* of P, and so on.

We obtain the *snub polytope* $S(P)$ of P by further modifying $E^2(P)$. The faces of $E^2(P)$ are of three kinds: the *first* correspond to the faces of P, the *second* correspond to the vertices of P, and the *third*, which are rectangles, correspond to the edges of P. In $S(P)$, each of these latter quadrangles is split into two triangles by a new edge; the splitting is done coherently (in one of two ways), so that the vertices of $S(P)$ are all 5-valent. Again, Steinitz's theorem guarantees the existence of $S(P)$ as a 3-polytope.

In the next lemma, we summarize some numerical facts, for subsequent reference.

LEMMA 12. *Let P be a 3-polytope. Then*:
(a) $f_0(E^2(P)) = 2f_1(P)$, $f_1(E^2(P)) = 4f_1(P)$, $f_2(E^2(P)) = f_0(P) + f_1(P) + f_2(P)$;
(b) $f_0(S(P)) = 2f_1(P)$, $f_1(S(P)) = 5f_1(P)$, $f_2(S(P)) = f_0(P) + 2f_1(P) + 2f_2(P)$.

In Fig. 4, we illustrate these various modifications, giving the Schlegel diagram of $E^2(P)$ and $S(P)$, where P is the prism over the triangle.

As an additional example to that of Fig. 4, if P is a tetrahedron, then $E(P)$ is an octahedron, $E^2(P)$ is a cuboctahedron, and $S(P)$ is an icosahedron.

We observe that both $E^2(P)$ and $S(P)$ have families of disjoint faces which cover the vertices, namely those of the first (or second) kind.

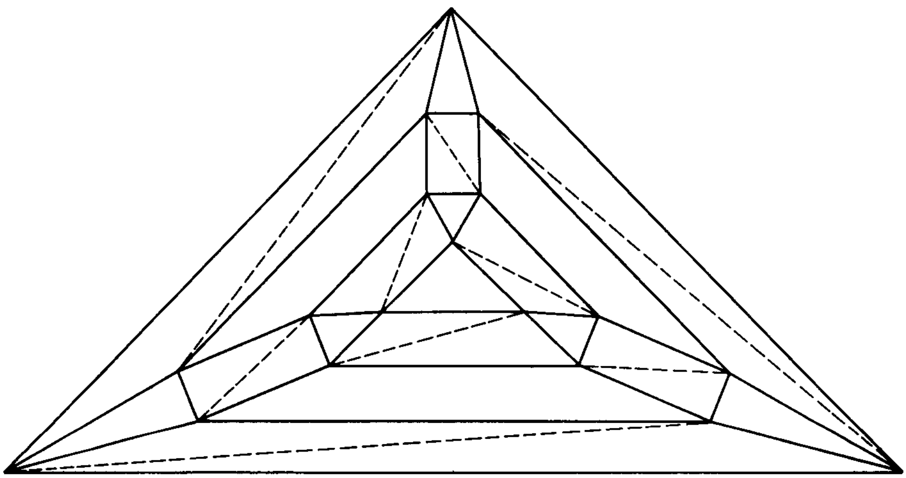

Fig. 4.

LEMMA 13. *If we apply methods* C *or* D *to the boundary complex of* $E^2(P)$ *or* $S(P)$, *using the faces of the first kind, then the resulting manifold* M *has*

$$f_0(M) = 4f_1(P), \qquad g(M) = f_0(P) - 1.$$

We now come to our applications.

LEMMA 14. *There is a* $\{3, 7; g\}$ *for each* $g \geqq 2$.

The case $g = 2$ is not covered by our general methods, and so must be treated separately.

So, first suppose that $g \geqq 3$. Let P be a simple 3-polytope with $g + 1$ faces (the tetrahedron for $g = 3$, and the prism over the $(g-1)$-gon for $g \geqq 4$ will do), and apply method D to $S(P)$, using the faces of $S(P)$ of the first kind. Then the resulting polyhedron M is in $\mathcal{M}_{3,7}$ (by Lemma 11(b)), and, by Lemma 13, $g(M) = g$. (As expected, $f_0(M) = 12(g-1)$.)

For $g = 2$, let N be Czaszar's torus, which is a $\{3, 6; 1\}$ with $f_0(N) = 7$. Thus every two vertices of N are joined by an edge ($f_1(N) = 21$). (For further details, see [9].) Let v be a vertex of N which is also a vertex of $\operatorname{conv} N$. Let the plane H support $\operatorname{conv} N$ in v alone, and perform a projectivity Π of E^3 taking H to infinity. Then $\Pi(N)$ has six parallel infinite edges. Let J be a plane cutting each of these edges (in an interior point) and perpendicular to them; thus the six (finite) vertices of $\Pi(N)$ lie to one side of J, say in the interior of the closed half-space J^+. Let Φ denote reflexion in J, and write $M_1 = (\Pi(N) \cap J^+) \cup \Phi(\Pi(N) \cap J^+)$. Then M_1 is a closed polyhedral 2-manifold with 12 vertices, each 6-valent. M_1 has 6 quadrangular faces, corresponding to the 6 infinite faces of $\Pi(N)$, which form a tube. Let M be obtained from M_1 by coherent splitting of these quadrangles into triangles, again as in Fig. 2, and moving the vertices into general position. Then $M \in \mathcal{M}_{3,7}$, and $g(M) = 2$, as required.

LEMMA 15. *There are* $\{4, 5; g\}$ *for* $g = 5, 7$ *and* $g \geqq 9$.

The case $g = 5$ is again somewhat different, so we deal with it later.

For $g = 7$ and $g \geqq 9$, let P be a 3-polytope with $g - 1$ 4-valent vertices. In fact, we must have $P = E(Q)$, for some 3-polytope Q, and Q can have only these numbers of edges. We now apply method C to $E^2(P)$, using the faces of $E^2(P)$ of the first kind. Using Lemma 11(a), we see that the resulting manifold M is in $\mathcal{M}_{4,5}$, and since $f_0(M) = 8(g - 1)$, we have $g(M) = g$, as required.

For $g = 5$, let N be a torus $\{4, 4; 1\}$ with 16 vertices, and let N' be an isomorphic torus inscribed in N, so that corresponding edges are parallel. For example, let N have vertices $(\pm 4, \pm 4, \pm 4)$ and $(\pm 1, \pm 1, \pm 1)$, and let N' have vertices $(\pm 3, \pm 3, \pm 2)$ and $(\pm 2, \pm 2, \pm 1)$. N has four disjoint faces covering its vertices, for example those with normal vector $(1, 0, 0)$. We now apply method C using these faces, and the corresponding parallel faces of N'. By Lemma 11(a), the resulting polyhedron M is in $\mathcal{M}_{4,5}$, and $f_0(M) = 32$, so that $g(M) = 5$, as required.

We may observe that analogous constructions yield all odd $g \geqq 5$.

LEMMA 16. *There are* $\{3, 8; g\}$ *for* $g \geqq 4$.

Let N be a torus $\{3, 6; 1\}$ with $3(g - 1)$ vertices, obtained by making the identifications shown in Fig. 5. The horizontal edges are identified in the natural way, and to identify the vertical edges we take remainders modulo 3. We can realize N geometrically as follows. Let T be an equilateral triangle inscribed in the unit circle in the plane $y = 0$ centred at $(10, 0, 0)$. Write $T_0 = T$, and let T_k

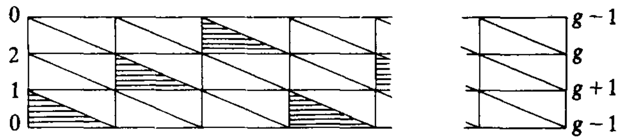

Fig. 5.

($k = 1, \cdots, n-1$) be obtained from T by rotating its plane about the z-axis through an angle $2k\pi/n$, and, if $n \not\equiv 0$ (modulo 3), additionally rotating T about its centre through an angle $2k\pi/3n$. Then we can take the vertices of N to be those of the T_k's. (Initially, we have coplanar adjacent faces, but we can deal with this at the end.) We similarly define N', taking $T' = 2T - (10,0,0)$, the homothetic copy of T of twice the size with the same centre. We now apply method D with the triangles of N hatched in Fig. 5, and move the vertices into general position if necessary. There results a polyhedron $M \in \mathcal{M}_{3,8}$ with $6(g-1)$ vertices, so that $g(M) = g$ as required.

§5. The method of intersecting octahedra

The following constructions are related to those for $\{6,4;g\}$ and $\{8,4;g\}$. However, whereas those could be described by a classical 4-dimensional technique, here we need a specifically 3-dimensional construction.

LEMMA 17. *There are* $\{5,4;g\}$ *for* $g = 5, 7$ *and* $g \geqq 9$.

As with the dual case $\{4,5;g\}$, the case $g = 5$ needs special treatment, which we give later.

So, suppose first that $g = 7$ or $g \geqq 9$. As in the proof of Lemma 15, let P be a 3-polytope with $g-1$ vertices, all 4-valent. We may suppose $0 \in \operatorname{int} P$. Let $\frac{1}{2} < \kappa < \frac{2}{3}$, $0 < \lambda < 1 < \mu$. Let v be any vertex of P, and let the four adjacent vertices be, in cyclic order around v, $v_1, \cdots, v_4$. Let Q_v be the (not necessarily convex) octahedron with pairs of opposite vertices λv, μv; $(1-\kappa)v + \kappa v_i$ ($i = 1,3$; $i = 2,4$). Let M be the boundary of $\bigcup\{Q_v \mid v \in \operatorname{vert} P\}$. Then $M \in \mathcal{M}_{5,4}$, and M has $8(g-1)$ faces, so $g(M) = g$ as required.

The only assertion that needs verifying is $M \in \mathcal{M}_{5,4}$. The condition $\kappa < \frac{2}{3}$ ensures that three octahedra Q_u, Q_v and Q_w cannot have a non-empty intersection, while the condition $\kappa > \frac{1}{2}$ and the choice of the vertices of the Q_v (so that appropriate sets of edges are coplanar) ensures that, if v and w are adjacent vertices, then $\operatorname{bd} Q_v \cap \operatorname{bd} Q_w$ is a quadrangle. Thus each original triangle of Q_v is truncated at two vertices, and so becomes a pentagon. Clearly now, four such pentagons meet at each vertex of M.

Now let $g = 5$. The manifold M we construct can also be regarded as an intersection of four suitable octahedra, but it is perhaps easier to describe it directly. M is symmetric by reflexion in each of the three coordinate planes, so it is enough to describe its four faces $F_1, \cdots, F_4$ in the non-negative orthant. The four F_i meet in a common vertex $V = (2,2,3)$, and $F_i \cap F_{i+1} = VW_i$ $(i = i + 4)$, where

$$W_1 = (0,3,2), \quad W_2 = (4,0,2), \quad W_3 = (0,5,2), \quad W_4 = (4,0,3).$$

Finally, F_i has a vertex X_i on the x-axis, and Y_i on the y-axis, where

$$X_1 = (1,0,0), \quad X_2 = (8/3,0,0), \quad X_3 = (24/5,0,0), \quad X_4 = (7,0,0),$$

$$Y_1 = (0,1,0), \quad Y_2 = (0,2,0), \quad Y_3 = (0,6,0), \quad Y_4 = (0,7,0).$$

We illustrate this in Fig. 6. We have $M \in \mathcal{M}_{5,4}$, since, by construction, all vertices are 4-valent, and $f_2(M) = 32$, whence follows $g(M) = 5$, as required.

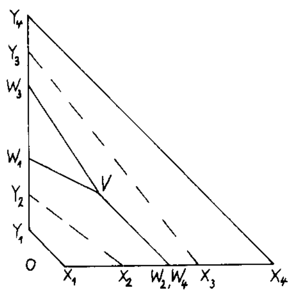

Fig. 6.

Note added in proof. A. Wanka (Siegen) recently constructed the equivelar polyhedra $\{4,5;g\}$, $g = 4,6,8$ and $\{5,4;g\}$, $g = 6,8$.

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